

Regime-switching diffusion processes: strong solutions and strong Feller property

Shao-Qin Zhang

School of Statistics and Mathematics, Central University of Finance and Economics, Beijing 100081, China

Email: zhangsq@mail.bnu.edu.cn

March 14, 2016

Abstract

We investigate the existence and uniqueness of strong solutions up to an explosion time for regime-switching diffusion processes in an infinite state space. Instead of concrete conditions on coefficients, our existence and uniqueness result is established under the general assumption that the diffusion in every fixed environment has a unique non-explosive strong solution. Moreover, non-explosion conditions for regime-switching diffusion processes are given. The strong Feller property is proved by further assuming that the diffusion in every fixed environment generates a strong Feller semigroup.

AMS Subject Classification (2010): 60H60, 60J05

Keywords: Regime-switching diffusions, Strong solutions, Strong Feller property

1 Introduction

Many uncertain hybrid systems from financial engineering, wireless communications, biology and etc. can be modeled by the regime-switching diffusion processes (RSDPs for short), see [20] and references. In these models, the continuous dynamics and discrete events coexist. Various properties of these processes have been concerned recently. For example, [5, 6, 8, 9, 11, 12, 19] had studied the recurrent properties, [2, 4, 12, 16, 20] concerned stability and optimal control, and [13] investigated the asymptotic properties. We mention monographs [4] and [20] for system surveys on the research of regime-switching processes.

Let \mathbb{S} be the set of all the natural numbers, i.e. $\mathbb{S} = \{1, 2, 3, \dots, n, \dots\}$, $(\Omega, \mathcal{F}, \mathbb{P})$ be a completed probability space. A regime-switching diffusion process is a two components process $\{(X_t, \Lambda_t)\}_{t \geq 0}$ described by

$$dX_t = b(X_t, \Lambda_t, t)dt + \sigma(X_t, \Lambda_t, t)dW_t, \quad (1.1)$$

and

$$\mathbb{P}(\Lambda_{t+\Delta t} = j \mid \Lambda_t = i, (X_s, \Lambda_s), s \leq t) = \begin{cases} q_{ij}(X_t)\Delta t + o(\Delta t), & i \neq j, \\ 1 + q_{ii}(X_t)\Delta t + o(\Delta t), & i = j, \end{cases} \quad (1.2)$$

where $\{W_t\}_{t \geq 0}$ is a Brownian motion on \mathbb{R}^d w.r.t a completed reference $\{\mathcal{F}_t\}_{t \geq 0}$ and

$$b : \mathbb{R}^d \times \mathbb{S} \times [0, \infty) \rightarrow \mathbb{R}^d, \quad \sigma : \mathbb{R}^d \times \mathbb{S} \times [0, \infty) \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d, \quad q_{ij} : \mathbb{R}^d \rightarrow \mathbb{R},$$

are measurable functions and

$$q_{ij}(x) \geq 0, \quad i \neq j, \quad \sum_{j \neq i} q_{ij}(x) \leq -q_{ii}(x).$$

The matrix $Q(x) = (q_{ij}(x))$ is called Q -matrix, see [1]. If $Q(x)$ is independent of x and Λ_t is independent of $\{W_t\}_{t \geq 0}$, $\{(X_t, \Lambda_t)\}_{t \geq 0}$ is called a state-independent RSDP, otherwise called a state-dependence RSDP. Let

$$q_i(x) = \sum_{j \neq i} q_{ij}(x), \quad i \in \mathbb{S}.$$

If $Q(x)$ is conservative, i.e. $-q_{ii}(x) = q_i(x)$, $x \in \mathbb{R}^d$, then as in [13, Chapter II-2.1] or [2, 19, 10], we can represent $\{(X_t, \Lambda_t)\}_{t \geq 0}$ in the form of a system of stochastic differential equations(SDEs for short) driven by $\{W_t\}$ and a Poisson random measure. Precisely, for each $x \in \mathbb{R}^d$, $\{\Gamma_{ij}(x) : i, j \in \mathbb{S}\}$ is a family of disjoint intervals on $[0, \infty)$ constructed as follows

$$\begin{aligned} \Gamma_{12}(x) &= [0, q_{12}(x)), \quad \Gamma_{13}(x) = [q_{12}(x), q_{12}(x) + q_{13}(x)), \dots \\ \Gamma_{21}(x) &= [q_1(x), q_1(x) + q_{21}(x)), \quad \Gamma_{23}(x) = [q_1(x) + q_{21}(x), q_1(x) + q_{21}(x) + q_{23}(x)), \dots \\ \Gamma_{31}(x) &= [q_1(x) + q_2(x), q_1(x) + q_2(x) + q_{13}(x)), \dots \\ &\dots \end{aligned}$$

We set $\Gamma_{ii}(x) = \emptyset$ and $\Gamma_{ij}(x) = \emptyset$ if $q_{ij}(x) = 0$ for $i \neq j$. Define a function $h : \mathbb{R}^d \times \mathbb{S} \times [0, \infty) \rightarrow \mathbb{R}$ as follows

$$h(x, i, z) = \sum_{j \in \mathbb{S}} (j - i) \mathbb{1}_{\Gamma_{ij}(x)}(z) = \begin{cases} j - i, & \text{if } z \in \Gamma_{ij}(x), \\ 0, & \text{otherwise.} \end{cases}$$

Let $N(dz, dt)$ be a Poisson random measure with intensity $dt dz$ and independent of the Brownian motion $\{W_t\}_{t \geq 0}$. Then we turn to concern the following equation

$$\begin{cases} dX_t = b(X_t, \Lambda_t, t)dt + \sigma(X_t, \Lambda_t, t)dW_t, \\ d\Lambda_t = \int_0^\infty h(X_{t-}, \Lambda_{t-}, z)N(dz, dt). \end{cases} \quad (1.3)$$

Denote the diffusion process in the fixed environment $i \in \mathbb{S}$ by x_t^i , see (2.1) for more precise definition. Showing as in (1.3) or (1.1) and (1.2), one can image that the diffusion component $\{X_t\}_{t \geq 0}$ is a hybrid process of $\{x_t^i\}_{i \in \mathbb{S}}$ via jump process Λ_t . A quite nature question is that if all the diffusion processes $\{\{x_t^i\}_{t \geq 0}\}_{i \in \mathbb{S}}$ process some comment property(such as pathwise uniqueness, strong Feller property, recurrence, ergodicity and etc.), then what about $\{(X_t, \Lambda_t)\}_{t \geq 0}$? In fact, many works, such as [9, 10, 12, 19], give us some insight on this problem.

There are some papers focusing on existence and uniqueness of strong solutions and strong Feller property for RSDPs. For instance, we refer to [14] for existence and uniqueness of solution in a finite state space, to [10, 15, 19] for strong Feller property.

It should be pointed out that, [10] established the existence and uniqueness and strong Feller property to the strong solution of state-dependent RSDPs in an infinite state space with some type of non-Lipschitz coefficients. However, existence and uniqueness theorem in [10] can not be applied to more irregular case, such as Hölder continuous coefficients or Sobolev coefficients, which have been intensively studied recently. Here, we only mention [17] for a brief survey in their introduction on strong uniqueness of SDEs with singular coefficients and references therein. Hence one aim of the this paper is to investigate the existence and uniqueness of the regime-switching processes that the coefficients may be singular. Another aim is to concern strong Feller property of $\{(X_t, \Lambda_t)\}_{t \geq 0}$ under the assumption that the diffusion in every fixed environment processes the same property. The method used here is different from [10]. Our argument is based on the following simple observation(formally), that is if $\{q_i\}_{i \in \mathbb{S}}$ are locally bounded and $\sup_{s \in [0, t]} (|X_{s-}| + \Lambda_{s-}) \leq M (M \in \mathbb{S})$, then

$$\bigcup_{s \in [0, t]} \text{supp}(h(X_{s-}, \Lambda_{s-}, \cdot)) \subset \bigcup_{|x| \leq M, i \leq M, j \geq 1} \Gamma_{i,j}(x) \subset \left[0, \sup_{|x| \leq M} \sum_{k=1}^{M+1} q_k(x)\right],$$

and $\sup_{|x| \leq M} \sum_{k=1}^{M+1} q_k(x) < \infty$. Let $K = [0, \sup_{|x| \leq M} \sum_{k=1}^{M+1} q_k(x)]$. Then

$$\Lambda_t = \Lambda_0 + \int_0^t \int_0^\infty h(X_{s-}, \Lambda_{s-}, z) N(dz, ds) = \Lambda_0 + \int_0^t \int_K h(X_{s-}, \Lambda_{s-}, z) N(dz, ds).$$

But on K , $N(K, t) := \int_0^t \int_K N(dz, ds)$ is a Poisson process which jumps finite times on any finite interval. This allows us to generalize the method used in [14](see also [3, Theorem IV.9.1]), and give an interlacing construction to the solution of (1.3) before (X_t, Λ_t) escapes a bounded domain. The strong Feller property is also studied following this idea which is also different from [10, 19], where they got the strong Feller property by a finite-state approximation argument. Since we do not assume the concrete conditions on coefficients of the diffusion part, our results can be applied to the case that the diffusion component may be irregular and degenerated.

The rest part of the paper is organized as follows. In Section 2, we discuss the existence, uniqueness and non-explosion of the solution to (1.3). In Section 3, we study the strong Feller property.

2 Existence, uniqueness and non-explosion

Throughout this paper, we assume that the Q -matrix $Q(x)$ is conservative, that is

$$-q_{ii}(x) = q_i(x) \left(= \sum_{j \neq i} q_{ij}(x) \right),$$

for more details on continuous time Markov chains, one can consult [1]. By this assumption, then we shall prove the existence of a unique strong solution to the equation (1.3).

We introduce the following hypothesis

Hypothesis (H) For each $i \in \mathbb{S}$, $t_0 \geq 0$, the following equation

$$dx_t = b(x_t, i, t + t_0)dt + \sigma(x_t, i, t + t_0)dW_t \quad (2.1)$$

has a unique non-explosive strong solution.

Let

$$W^d = C([0, \infty), \mathbb{R}^d), \quad W_0^d = \left\{ \omega \in W^d \mid \omega(0) = 0 \right\},$$

with the topology of locally uniformly convergence on $[0, \infty)$, $\mathcal{B}(W^d)$ be the Borel σ -field of W^d and $\mathcal{B}_t(W^d)$ be the sub- σ -field of $\mathcal{B}(W^d)$ generated by $\omega(s)$, $0 \leq s \leq t$. We recall that, for each $(i, t_0) \in \mathbb{S} \times [0, \infty)$, a d dimensional continuous process $x^{i, t_0} := \{x_t^{i, t_0}\}_{t \geq 0}$ defined on a probability space with a reference family and a Brownian motion $\{W_t\}_{t \geq 0}$ is a strong solution of (2.1), if there is a function $F(i, t_0, \cdot, \cdot) : \mathbb{R}^d \times W_0^d \rightarrow W^d$ such that

- (1) for all Borel probability measure μ on \mathbb{R}^d , there is a function $\tilde{F}_\mu(i, t_0, \cdot, \cdot) : \mathbb{R}^d \times W_0^d \rightarrow W^d$ which is $\overline{\mathcal{B}(\mathbb{R}^d \times W_0^d)}^{\mu \times \mathbb{P}^W} / \mathcal{B}(W^d)$ -measurable, and for μ -a.e. $x \in \mathbb{R}^d$ it holds that

$$F(i, t_0, x, \omega) = \tilde{F}_\mu(i, t_0, x, \omega), \quad \mathbb{P}^W\text{-a.s. } \omega \in W_0^d,$$

where \mathbb{P}^W is the Wiener measure on W_0^d ,

- (2) for each $x \in \mathbb{R}^d$, $F(i, t_0, x, \cdot) : W_0^d \rightarrow W^d$ is $\overline{\mathcal{B}_t(W_0^d)}^{\mathbb{P}^W} / \mathcal{B}_t(W^d)$ measurable for each $t \geq 0$, and $x^{i, t_0} = F(i, t_0, x_0^{i, t_0}, W)$.

For more details on the strong solution of SDEs, we refer to [3, Chapter IV. Definition 1.6]. In the following, we shall denote the solution of (2.1) by x_t^{x, i, t_0} if it starts from $x \in \mathbb{R}^d$ with parameters i, t_0 in the coefficients, and omit t_0 in superscript if $t_0 = 0$. Here, we emphasize that in (2), the initial value of x_t^{i, t_0} , i.e. x_0^{i, t_0} , can be chosen arbitrary though we write parameters i, t_0 in superscript.

In general, the solution to (1.3) can be explosive. So we consider the local solution.

Definition 2.1. An adapted process $\{(X_t, \Lambda_t)\}_{t \in [0, \tau)}$ with the first component is continuous and the second one is cadlag is called a local solution of (1.3) with life time τ , if $\tau > 0$ is a stopping time such that \mathbb{P} -a.s. $\limsup_{t \uparrow \tau} (|X_t| + \Lambda_t) = \infty$ holds on $\{\tau < \infty\}$ and \mathbb{P} -a.s.

$$\begin{cases} X_t = X_0 + \int_0^t b(X_s, \Lambda_s, s) ds + \int_0^t \sigma(X_s, \Lambda_s, s) dW_s, \\ \Lambda_t = \Lambda_0 + \int_0^t \int_0^\infty h(X_s, \Lambda_{s-}, z) N(dz, ds), \quad t \in [0, \tau). \end{cases}$$

Our first result is the following

Theorem 2.1. Assume that (H) holds, and for each $k \in \mathbb{S}$, q_k is locally bounded. Then for each initial value $(x, i) \in \mathbb{R}^d \times \mathbb{S}$, the equation (1.3) has a unique local strong solution $\{(X_t, \Lambda_t)\}_{t \in [0, \tau)}$ with life time τ .

We can extend Theorem 2.1 to some cases that the diffusion in each environment may be explosive.

Corollary 2.2. For all $M > 0$, let

$$b_M(x, i, t) = b(x, i, t) \mathbb{1}_{\{|x| \leq M, t \leq M\}}(x), \quad \sigma_M(x, i, t) = \sigma(x, i, t) \mathbb{1}_{\{|x| \leq M, t \leq M\}}(x).$$

If for all $M > 0$, $i \in \mathbb{S}$, the equation

$$dx_t^M = b_M(x_t^M, i, t)dt + \sigma_M(x_t^M, i, t)dW_t$$

has a unique non-explosive strong solution, and $q_i(\cdot)$ is locally bounded, then the equation (1.3) has a unique local strong solution $\{(X_t, \Lambda_t)\}_{t \in [0, \tau)}$ with lift time τ .

Before the proof of our results, we present a lemma on a SDE (see (2.2)) with random coefficients related to (2.1). Let $(\tilde{\mathbb{P}}, \tilde{\Omega}, \tilde{\mathcal{F}})$ be some completed probability space, $\{\tilde{W}_t\}_{t \geq 0}$ be a Brownian motion on this probability space w.r.t. some completed reference $\{\tilde{\mathcal{F}}_t\}_{t \geq 0}$. Let $(\eta, \zeta, \xi) : \tilde{\Omega} \rightarrow \mathbb{S} \times (0, \infty) \times \mathbb{R}^d$ be a random vector which is independent of $\{\tilde{W}_t\}_{t \geq 0}$ and $(\eta, \zeta, \xi) \in \tilde{\mathcal{F}}_0$. Denote the distribution of (η, ζ, ξ) by μ . Let

$$\mathcal{G}_t = \sigma \left(\left\{ \eta, \zeta, \xi, \tilde{W}_s \mid s \leq t \right\} \right),$$

\mathcal{G}_t be the completion of \mathcal{G}_t by $\tilde{\mathbb{P}}$. Consider the following SDEs with random coefficients

$$dx_t = b(x_t, \eta, t + \zeta)dt + \sigma(x_t, \eta, t + \zeta)d\tilde{W}_t, \quad x_0 = \xi. \quad (2.2)$$

We say a continuous adapted process $\{x_t\}_{t \geq 0}$ is a solution to (2.2) with a Brownian motion $\{\tilde{W}_t\}_{t \geq 0}$, if

$$\int_0^t |b(x_s, \eta, s + \zeta)| + |\sigma(x_s, \eta, s + \zeta)|^2 ds < \infty, \quad t \geq 0, \quad \tilde{\mathbb{P}}\text{-a.s.}$$

and

$$x_t = x_0 + \int_0^t b(x_s, \eta, s + \zeta)ds + \int_0^t \sigma(x_s, \eta, s + \zeta)d\tilde{W}_s, \quad t \geq 0, \quad \tilde{\mathbb{P}}\text{-a.s.}$$

Due to the hypothesis **(H)**, there exists $F : \mathbb{S} \times (0, \infty) \times \mathbb{R}^d \times W_0^d \rightarrow W^d$ such that $\{F(i, t_0, x, \tilde{W}.)\}_{t \geq 0}$ is the solution of (2.1) starting from x . Since (η, ζ, ξ) is independent of $\{\tilde{W}_t\}_{t \geq 0}$ and $(\xi, \eta, \zeta) \in \tilde{\mathcal{F}}_0$, $\{F(\xi, \eta, \zeta, \tilde{W}.)\}_{t \geq 0}$ is a solution of (2.2).

Lemma 2.3. *Assume that **(H)** holds. Let $\{x'_t\}_{t \geq 0}$ be a solution of (2.2). Then $x'_t = F(\eta, \zeta, \xi, \tilde{W}.)_t$, $t \geq 0$, $\tilde{\mathbb{P}}$ -a.s.*

Proof. It is clear that $x' \in \mathcal{G}_\infty$ and $x'_t \in \mathcal{G}_t$, so there is $F'_\mu : \mathbb{S} \times (0, \infty) \times \mathbb{R}^d \times W_0^d \rightarrow W^d$ such that

$$F'_\mu(\eta, \zeta, \xi, \tilde{W}.)_t \in \tilde{\mathcal{F}}_t, \quad \text{and} \quad \tilde{\mathbb{P}} \left(x' = F'_\mu(\eta, \zeta, \xi, \tilde{W}.) \right) = 1.$$

Due to [7, Lemma V.10.1], there is $\Phi : \mathbb{S} \times (0, \infty) \times \mathbb{R}^d \times W_0^d \rightarrow W^d$ such that

$$\int_0^t \sigma(F'_\mu(\eta, \zeta, \xi, \tilde{W}.)_s, \eta, s + \zeta)d\tilde{W}_s = \Phi(\eta, \zeta, \xi, \tilde{W}.)_t, \quad t \geq 0.$$

Then we have $\tilde{\mathbb{P}}$ -a.s.

$$F'_\mu(\eta, \zeta, \xi, \tilde{W}.)_t = \xi + \Phi(\eta, \zeta, \xi, \tilde{W}.)_t + \int_0^t b(F'_\mu(\eta, \zeta, \xi, \tilde{W}.)_s, \eta, s + \zeta)ds, \quad t \geq 0.$$

Hence, for μ -a.s. $(i, t_0, x) \in \mathbb{S} \times (0, \infty) \times \mathbb{R}^d$, $\tilde{\mathbb{P}}$ -a.s.

$$F'_\mu(i, t_0, x, \tilde{W}.)_t = x + \Phi(i, t_0, x, \tilde{W}.)_t + \int_0^t b(F'_\mu(i, t_0, x, \tilde{W}.)_s, i, s + t_0) ds, \quad t \geq 0.$$

That means $F'_\mu(i, t_0, x, \tilde{W}.)_t$ is a solution of (2.1) with W replaced by \tilde{W} . If

$$\tilde{\mathbb{P}}(x' \neq F(\eta, \zeta, \xi, \tilde{W}.)_t) > 0,$$

then

$$\tilde{\mathbb{P}}(F(\eta, \zeta, \xi, \tilde{W}.)_t \neq F'_\mu(\eta, \zeta, \xi, \tilde{W}.)_t) > 0.$$

Since (η, ζ, ξ) is independent of $\{\tilde{W}_t\}_{t \geq 0}$, there is $A \subset \mathbb{S} \times (0, \infty) \times \mathbb{R}^d$ with $\mu(A) > 0$ such that

$$\tilde{\mathbb{P}}(F(i, t_0, x, \tilde{W}.)_t \neq F'_\mu(i, t_0, x, \tilde{W}.)_t) > 0, \quad (i, t_0, x) \in A.$$

It contradicts **(H)**, since $\{F(i, t_0, x, \tilde{W}.)_t\}_{t \geq 0}$ and $\{F'_\mu(i, t_0, x, \tilde{W}.)_t\}_{t \geq 0}$ are two solutions of (2.1) on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with the same Brownian motion \tilde{W} . □

Proof of Theorem 2.1

Let $K > 0$. We also denote $K = [0, K]$ without confusion. We first show the existence and uniqueness of solutions to the following equation

$$\begin{cases} dX_t^K = b(X_t^K, \Lambda_t^K, t)dt + \sigma(X_t^K, \Lambda_t^K, t)dW_t, & X_0^K = x \\ d\Lambda_t^K = \int_K h(X_{t-}^K, \Lambda_{t-}^K, z)N(dz, dt), & \Lambda_0^K = i. \end{cases} \quad (2.3)$$

Let

$$N(K, t) = \int_0^t \int_K N(dz, ds), \quad N_t^K = \int_0^t \int_K zN(dz, ds).$$

Then $\{N(K, t)\}_{t \geq 0}$ is a Poisson process and $\{N_t^K\}_{t \geq 0}$ is a compound Poisson process. Let

$$T_1^K = \inf\{t > 0 \mid N(K, t) > 0\}, \quad T_{n+1}^K = \inf\{t > T_n^K \mid N(K, t) > n\}, \quad n \geq 1.$$

Denote the size of the n -th jump of N_t^K by ΔN_n^K . Then $\{T_n^K\}_{n \in \mathbb{N}}$, $\{\Delta N_n^K\}_{n \in \mathbb{N}}$ and $\{W_t\}_{t \geq 0}$ are independent of each other and

$$0 < T_1^K < T_2^K < \dots < T_n^K < \dots < \infty.$$

Set $(X_t^K, \Lambda_t^K) = (x_t^{x, i, 0}, i)$, $t \in [0, T_1^K)$, and $X_{T_1^K}^K = x_{T_1^K}^{x, i, 0}$. Hence X_t^K satisfies that

$$\begin{aligned} X_{t \wedge T_1^K}^K &= x + \int_0^{t \wedge T_1^K} b(x_t^{x, i, 0}, i, s)dt + \int_0^{t \wedge T_1^K} \sigma(x_t^{x, i, 0}, i, s)dW_s \\ &= x + \int_0^{t \wedge T_1^K} b(X_s^K, \Lambda_s^K, s)ds + \int_0^{t \wedge T_1^K} \sigma(X_s^K, \Lambda_s^K, s)dW_s. \end{aligned}$$

Then at T_1^K , we have

$$\Lambda_{T_1^K} = i + h(X_{T_1^K-}^K, i, \Delta N_1^K) = i + h(x_{T_1^K-}^{x,i,0}, i, \Delta N_1^K).$$

So, according to **(H)**, the solution to (2.3) on $[0, T_1^K]$ exists uniquely and (X_t^K, Λ_t^K) can be constructed as above.

Next, set $X_0^{1,K} = X_{T_1^K}^K$, $\Lambda_0^{1,K} = \Lambda_{T_1^K}$ and

$$W_t^1 = W_{t+T_1^K} - W_{T_1^K}, \quad N_t^1 = N_{T_1^K+t}^K - N_{T_1^K}^K, \quad t \geq 0.$$

According to the strong Markov property, $\{W_t^1\}_{t \geq 0}$ is also a Brownian motion and N_t^1 is compound Poisson process w.r.t $\{\mathcal{F}_{T_1^K+t}^K\}_{t \geq 0}$. Let $N^1(dz, dt)$ be the corresponding Poisson random measure of N_t^1 . We consider

$$\begin{cases} X_t^{1,K} = X_{T_1^K}^K + \int_0^t b(X_s^{1,K}, \Lambda_s^{1,K}, s + T_1^K) ds + \sigma(X_s^{1,K}, \Lambda_s^{1,K}, s + T_1^K) dW_s^1, \\ \Lambda_t^{1,K} = \Lambda_{T_1^K} + \int_0^t \int_K h(X_{s-}^{1,K}, \Lambda_{s-}^{1,K}, z) N^1(dz, ds). \end{cases} \quad (2.4)$$

Denote the time of the first arrival of $N^1(K, t)$ by $T_1^{1,K}$, then $T_1^{1,K}$ coincide with $T_2^K - T_1^K$. It is clear that we can set

$$\Lambda_t^{1,K} = \Lambda_0^{1,K}, \quad t < T_1^{1,K}.$$

Then $X_t^{1,K}$ should satisfy

$$X_t^{1,K} = X_0^{1,K} + \int_0^t b(X_s^{1,K}, \Lambda_0^{1,K}, s + T_1^K) ds + \int_0^t \sigma(X_s^{1,K}, \Lambda_0^{1,K}, s + T_1^K) dW_s^1, \quad t < T_1^{1,K},$$

with $\{W_s^1\}_{s \geq 0}$ is independent of $\mathcal{F}_{T_1^K}$ and $\Lambda_0^{1,K}, T_1^K \in \mathcal{F}_{T_1^K}$. So, according to Lemma 2.3, (2.4) has a unique solution on $[0, T_1^{1,K}]$ and we can construct it as follows

$$X_t^{1,K} := F(X_0^{1,K}, \Lambda_0^{1,K}, T_1^K, W^1)_t, \quad 0 < t < T_1^{1,K}.$$

Then

$$\Lambda_{T_1^{1,K}} = \Lambda_0^{1,K} + h(X_{T_1^{1,K}-}^{1,K}, \Lambda_0^{1,K}, \Delta N_1^K).$$

Set $X_{T_1^{1,K}}^{1,K} = F(X_0^{1,K}, \Lambda_0^{1,K}, T_1^K, W^1)_{T_1^{1,K}}$. Define

$$(X_t^K, \Lambda_t^K) = (X_{t-T_1^K}^{1,K}, \Lambda_{t-T_1^K}^{1,K}), \quad t \in [T_1^K, T_2^K].$$

By continuing this process, we can determine the solution of (2.3) on all $[0, T_n^K]$, $n \geq 1$. Since $T_n^K \rightarrow \infty$ as $n \rightarrow \infty$ and Hypothesis **(H)**, we obtain a unique non-explosive solution (X_t^K, Λ_t^K) to (2.3).

Next, let $M, K \in \mathbb{S}$ such that

$$M > |x| + i, \quad K \geq \sup_{|y| \leq M} \sum_{k=1}^{M+1} q_k(y). \quad (2.5)$$

Set

$$\tau_M^K = \inf\{t \geq 0 \mid |X_t^K| + \Lambda_t^K \geq M\}, \quad \tau_M^{K+1} = \inf\{t \geq 0 \mid |X_t^{K+1}| + \Lambda_t^{K+1} \geq M\},$$

where $(X_t^{K+1}, \Lambda_t^{K+1})$ is the solution of (2.3) with K replaced by $K+1$. Since $T_1^K > 0$, \mathbb{P} -a.s. and

$$(X_t^K, \Lambda_t^K) = (x_t^{x_0, i}, i), \quad t < T_1^K,$$

we have $\tau_M^K > 0$, \mathbb{P} -a.s. Similarly, $\tau_M^{K+1} > 0$, \mathbb{P} -a.s. Note that

$$\begin{aligned} \text{supp} \left(h(X_{t-}^{K+1}, \Lambda_{t-}^{K+1}, \cdot) \right) &\subset \bigcup_{|x| \leq M, i \leq M, j \geq 1} \Gamma_{i,j}(x) \\ &\subset \left[0, \sup_{|x| \leq M} \sum_{k=1}^{M+1} q_k(x) \right] \subset [0, K], \quad t \leq \tau_M^{K+1}. \end{aligned}$$

Then

$$\begin{aligned} X_t^{K+1} &= x + \int_0^t b(X_s^{K+1}, \Lambda_s^{K+1}, s) ds + \int_0^t \sigma(X_s^{K+1}, \Lambda_s^{K+1}, s) dW_s, \\ \Lambda_t^{K+1} &= i + \int_0^t \int_{K+1} h(X_{s-}^{K+1}, \Lambda_{s-}^{K+1}, z) N(dz, ds) \\ &= i + \int_0^t \int_K h(X_{s-}^{K+1}, \Lambda_{s-}^{K+1}, z) N(dz, ds), \quad t \leq \tau_M^{K+1}. \end{aligned}$$

Due to the uniqueness of (2.3), we get that $\tau_M^{K+1} \leq \tau_M^K$ and

$$(X_t^{K+1}, \Lambda_t^{K+1}) = (X_t^K, \Lambda_t^K), \quad t \leq \tau_M^{K+1}.$$

However,

$$|X_{\tau_M^{K+1}}^{K+1}| + \Lambda_{\tau_M^{K+1}}^{K+1} \geq M,$$

so, in fact, we have $\tau_M^{K+1} = \tau_M^K$. Hence, we set

$$\tau_M = \lim_{K \rightarrow \infty} \tau_M^K, \quad \tau = \lim_{M \rightarrow \infty} \tau_M.$$

τ_M is an increasing sequence of stopping time and $\tau_M > 0$. Now, for each $M \in \mathbb{S}$ with $M > |x| + i$ and K large enough (recalling that (2.5)), we define

$$(X_t, \Lambda_t) = (X_t^K, \Lambda_t^K), \quad t < \tau_M = \tau_M^K.$$

Then (X_t, Λ_t) satisfies (1.3) for $t < \tau$ and

$$\limsup_{t \uparrow \tau} (|X_t| + \Lambda_t) = \infty.$$

Lastly, we prove the pathwise uniqueness of solutions to (1.3). Let (X_t, Λ_t) and $(\tilde{X}_t, \tilde{\Lambda}_t)$ be two solutions in the same probability space with the same Brownian motion and Poisson random measure. For each $M \in \mathbb{S}$ with $M > |x| + i$, let

$$\tau_M = \inf \left\{ t \geq 0 \mid |X_t| + |\Lambda_t| \geq M \right\}, \quad \tilde{\tau}_M = \inf \left\{ t \geq 0 \mid |\tilde{X}_t| + |\tilde{\Lambda}_t| \geq M \right\}. \quad (2.6)$$

Let $K = \left[0, \sup_{|x| \leq M} \sum_{k=1}^{M+1} q_k(x) \right]$. Then

$$\begin{cases} X_t = x + \int_0^t b(X_s, \Lambda_s, s) ds + \int_0^t \sigma(X_s, \Lambda_s, s) dW_s, \\ \Lambda_t = i + \int_0^t \int_K h(X_{s-}, \Lambda_{s-}, z) N(dz, ds), \quad t \leq \tau_M. \end{cases} \quad (2.7)$$

That means (X_t, Λ_t) is a solution of (2.3) before τ_M . So does for $(\tilde{X}_t, \tilde{\Lambda}_t)$. Then by uniqueness of (2.3), we obtain that

$$(X_{t \wedge \tau_M}, \Lambda_{t \wedge \tau_M}) = (\tilde{X}_{t \wedge \tilde{\tau}_M}, \tilde{\Lambda}_{t \wedge \tilde{\tau}_M}).$$

Therefore we get the pathwise uniqueness of solutions to (1.3). \square

Proof of Corollary 2.2

We adapt the stopping time skill used in [18, Theorem 1.1]. Let $M \in \mathbb{S}$ such that $M > |x| + i$. We first consider the following equation

$$\begin{cases} dX_t^M = b_M(X_t^M, \Lambda_t^M, t)dt + \sigma_M(X_t^M, \Lambda_t^M, t)dW_t, X_0^M = x, \\ d\Lambda_t^M = \int_0^\infty h(X_{t-}^M, \Lambda_{t-}^M, z)N(dz, dt), \Lambda_0^M = i. \end{cases} \quad (2.8)$$

By Theorem 2.1 and our assumption, (2.8) has a unique local strong solution. Let $(X_t^{M+1}, \Lambda_t^{M+1})$ be the solution of (2.8) with M replaced by $M+1$,

$$\begin{aligned} \tau_M^M &= \inf\{t \in [0, M) \mid |X_t^M| + \Lambda_t^M \geq M\}, \\ \tau_M^{M+1} &= \inf\{t \in [0, M) \mid |X_t^{M+1}| + \Lambda_t^{M+1} \geq M\}. \end{aligned}$$

Since $|X_{t \wedge \tau_M^{M+1}}| \leq M$, we have that

$$\begin{aligned} X_{t \wedge \tau_M^{M+1}}^{M+1} &= x + \int_0^{t \wedge \tau_M^{M+1}} b_{M+1}(X_s^{M+1}, \Lambda_s^{M+1}, s)ds \\ &\quad + \int_0^{t \wedge \tau_M^{M+1}} \sigma_{M+1}(X_s^{M+1}, \Lambda_s^{M+1}, s)dW_s \\ &= x + \int_0^{t \wedge \tau_M^{M+1}} b_M(X_s^{M+1}, \Lambda_s^{M+1}, s)ds \\ &\quad + \int_0^{t \wedge \tau_M^{M+1}} \sigma_M(X_s^{M+1}, \Lambda_s^{M+1}, s)dW_s, \\ \Lambda_{t \wedge \tau_M^{M+1}}^{M+1} &= i + \int_0^{t \wedge \tau_M^{M+1}} \int_0^\infty h(X_s^{M+1}, \Lambda_{s-}^{M+1}, z)N(dz, ds). \end{aligned}$$

By pathwise uniqueness of (2.8), $(X_{t \wedge \tau_M^M}^M, \Lambda_{t \wedge \tau_M^M}^M) = (X_{t \wedge \tau_M^{M+1}}^{M+1}, \Lambda_{t \wedge \tau_M^{M+1}}^{M+1})$, for all $t \geq 0$, \mathbb{P} -a.s. Then we get that $\tau_{M+1}^{M+1} \geq \tau_M^{M+1} \geq \tau_M^M$. That means $\{\tau_k^k\}_{k \geq M, k \in \mathbb{S}}$ is a sequence of stopping time and increasing. Let $\tau = \lim_{k \rightarrow \infty} \tau_k^k$ and

$$(X_t, \Lambda_t) = (X_t^k, \Lambda_t^k), t \leq \tau_k^k.$$

Then $\{(X_t, \Lambda_t)\}_{t \in [0, \tau)}$ is a local solution of (1.3). The uniqueness can be prove similarly as Theorem 2.1. \square

We devote next theorem to the non-explosion of $\{(X_t, \Lambda_t)\}_{t \in [0, \tau)}$.

Theorem 2.4. Fix $T > 0$. (X_t, Λ_t) is a local solution of (1.3) starting from (x, i) . Assume that

$$\int_0^T \sup_{|y|+j \leq R} |\sigma(y, j, t)|_{HS}^2 dt < \infty, \quad R > 0, \quad (2.9)$$

and there is a constant $\beta > 0$ such that the function

$$(y, j) \in \mathbb{R}^d \times \mathbb{S} \mapsto \sum_{k=1}^{\infty} |k^\beta - j^\beta| q_{jk}(y) \quad (2.10)$$

is locally bounded on $\mathbb{R}^d \times \mathbb{S}$. If b, σ, q_{ij} satisfy one of the following conditions,

- (1) there exist a constant $p \geq 1$ and a non-negative function $C \in L^1([0, T], \mathbb{R}^+)$ such that

$$\begin{aligned} & \frac{\sum_{k=1}^{\infty} (k^\beta - j^\beta) q_{jk}(y)}{(1 + |y|^2)^p} + \frac{2\langle y, b(y, j, t) \rangle + (2p - 1)|\sigma(y, j, t)|_{HS}^2}{1 + |y|^2} \\ & \leq C(t) \left[1 + \frac{j^\beta}{(1 + |y|^2)^p} \right], \quad (y, j) \in \mathbb{R}^d \times \mathbb{S}, \end{aligned}$$

- (2) there exist $\alpha \in (0, 1]$, $c > 0$ such that

$$\begin{aligned} & \frac{2\langle y, b(y, j, t) \rangle + (1 + 2\alpha(1 + |y|^2)^\alpha) |\sigma(y, j, t)|_{HS}^2}{1 + |y|^2} \\ & + \frac{\sum_{k \leq j}^{\infty} (k^\beta - j^\beta) q_{jk}(y)}{(1 + |y|^2)^\alpha \exp \{(1 + |y|^2)^\alpha\}} + \frac{\sum_{k > j}^{\infty} (k^\beta - j^\beta) q_{jk}(y)}{(1 + |y|^2)^\alpha \exp \{e^{-\alpha c T} (1 + |y|^2)^\alpha\}} \\ & \leq c \left[1 + \frac{j^\beta}{(1 + |y|^2)^\alpha \exp \{e^{-\alpha c T} (1 + |y|^2)^\alpha\}} \right], \quad (y, j) \in \mathbb{R}^d \times \mathbb{S}, \end{aligned} \quad (2.11)$$

then the solution (X_t, Λ_t) is non-explosive on $[0, T]$.

Proof. Let τ_M as in (2.6). If (1) holds, then by Itô's formula, we have

$$\begin{aligned} & d(1 + |X_t|^2)^p \\ & = p(1 + |X_t|^2)^{p-1} \left\{ \left[2\langle X_t, b(X_t, \Lambda_t, t) \rangle + |\sigma(X_t, \Lambda_t, t)|_{HS}^2 \right. \right. \\ & \quad \left. \left. + \frac{2(p-1)|\sigma^*(X_t, \Lambda_t, t)X_t|^2}{1 + |X_t|^2} \right] dt + 2\langle X_t, \sigma(X_t, \Lambda_t, t)dW_t \rangle \right\} \\ & \leq p(1 + |X_t|^2)^{p-1} \left\{ \left[2\langle X_t, b(X_t, \Lambda_t, t) \rangle + (2p-1)|\sigma(X_t, \Lambda_t, t)|_{HS}^2 \right] dt \right. \\ & \quad \left. + 2\langle X_t, \sigma(X_t, \Lambda_t, t)dW_t \rangle \right\}, \quad t < \tau_M. \end{aligned} \quad (2.12)$$

Moreover, by (2.9)

$$\begin{aligned} & \mathbb{E} \int_0^t (1 + |X_{s \wedge \tau_M}|^2)^{p-1} |\sigma^*(X_{s \wedge \tau_M}, \Lambda_{s \wedge \tau_M-}, s) X_{s \wedge \tau_M}|^2 ds \\ & \leq M^2 (1 + M^2)^{p-1} \int_0^t \sup_{|x|+j \leq M} |\sigma(x, j, s)|_{HS}^2 ds \end{aligned}$$

$$< \infty.$$

Hence, $\left\{ \int_0^t (1 + |X_s|^2)^{p-1} \langle X_s, \sigma(X_s, \Lambda_s, s) dW_s \rangle \right\}_{t \geq 0}$ is a local martingale. On the other hand, since

$$\text{supp}(h(X_{t-}, \Lambda_{t-}, \cdot)) \subset \bigcup_{|x| \leq M, i \leq M, j \geq 1} \Gamma_{i,j}(x) \subset \left[0, \sup_{|x| \leq M} \sum_{k=1}^{M+1} q_k(x) \right], \quad t \leq \tau_M,$$

we have that $\{\Lambda_t\}_{t \in [0, \tau_M]}$ processes finite many jumps and

$$\Lambda_t = i + \int_0^t \int_{[0, K]} h(X_{s-}, \Lambda_{s-}, z) N(dz, ds), \quad t \leq \tau_M.$$

So, by Itô's formula, we obtain that

$$\begin{aligned} \Lambda_{t \wedge \tau_M}^\beta &= i^\beta + \int_0^{t \wedge \tau_M} \int_{\mathbb{R}^+} \left[(\Lambda_{s-} + h(X_{s-}, \Lambda_{s-}, z))^\beta - \Lambda_{s-}^\beta \right] N(dz, ds) \\ &= i^\beta + \int_0^{t \wedge \tau_M} \int_{\mathbb{R}^+} \left[(\Lambda_{s-} + h(X_{s-}, \Lambda_{s-}, z))^\beta - \Lambda_{s-}^\beta \right] \tilde{N}(dz, ds) \\ &\quad + \int_0^{t \wedge \tau_M} \int_{\mathbb{R}^+} \left[(\Lambda_{s-} + h(X_{s-}, \Lambda_{s-}, z))^\beta - \Lambda_{s-}^\beta \right] dz ds \\ &= i^\beta + \int_0^{t \wedge \tau_M} \int_{\mathbb{R}^+} \left[(\Lambda_{s-} + h(X_{s-}, \Lambda_{s-}, z))^\beta - \Lambda_{s-}^\beta \right] \tilde{N}(dz, ds) \quad (2.13) \\ &\quad + \int_0^{t \wedge \tau_M} \int_{\mathbb{R}^+} \sum_{k=1}^{\infty} \left[k^\beta - \Lambda_{s-}^\beta \right] \mathbb{1}_{[h(X_{s-}, \Lambda_{s-}, z) + \Lambda_{s-} = k]}(z) dz ds \\ &= i^\beta + \int_0^{t \wedge \tau_M} \int_{\mathbb{R}^+} \left[(\Lambda_{s-} + h(X_{s-}, \Lambda_{s-}, z))^\beta - \Lambda_{s-}^\beta \right] \tilde{N}(dz, ds) \\ &\quad + \int_0^{t \wedge \tau_M} \sum_{k=1}^{\infty} \left[k^\beta - \Lambda_{s-}^\beta \right] q_{\Lambda_{s-} - k}(X_{s-}) ds. \end{aligned}$$

Moreover, since (2.10) is locally bounded, we obtain that

$$\begin{aligned} \mathbb{E} \int_0^t \int_{\mathbb{R}^+} \left| (\Lambda_{s \wedge \tau_M-} + h(X_{s \wedge \tau_M-}, \Lambda_{s \wedge \tau_M-}, z))^\beta - \Lambda_{s \wedge \tau_M-}^\beta \right| dz ds \\ \leq \mathbb{E} \int_0^t \sum_{k=1}^{\infty} \left| k^\beta - \Lambda_{s-}^\beta \right| q_{\Lambda_{s \wedge \tau_M-} - k}(X_{s \wedge \tau_M-}) ds \\ \leq \int_0^t \sup_{|x| + j \leq M} \sum_{k=1}^{\infty} \left| k^\beta - j^\beta \right| q_{jk}(x) ds \\ < \infty. \end{aligned}$$

Hence, $\left\{ \int_0^t \int_{\mathbb{R}^+} \left[(\Lambda_{s \wedge \tau_M-} + h(X_{s \wedge \tau_M-}, \Lambda_{s \wedge \tau_M-}, z))^\beta - \Lambda_{s \wedge \tau_M-}^\beta \right] \tilde{N}(dz, ds) \right\}_{t \geq 0}$ is a martingale and $\mathbb{E} \Lambda_{t \wedge \tau_M}^\beta < \infty$. Then, following from (2.12) and (2.13), we get that

$$\mathbb{E} \left[(1 + |X_{t \wedge \tau_M}|^2)^p + p \Lambda_{t \wedge \tau_M}^\beta \right] - (1 + |x|^2)^p - p i^\beta$$

$$\begin{aligned}
&\leq \mathbb{E} \int_0^{t \wedge \tau_M} \left\{ p(1 + |X_s|^2)^{p-1} \left[2\langle X_s, b(X_s, \Lambda_s, s) \rangle + (2p-1) |\sigma(X_s, \Lambda_s, s)|_{HS}^2 \right] \right. \\
&\quad \left. + \sum_{k=1}^{\infty} \left[k^\beta - \Lambda_{s-}^\beta \right] q_{\Lambda_{s-k}}(X_{s-}) \right\} ds \\
&\leq \mathbb{E} \int_0^{t \wedge \tau_M} C(s) \left[(1 + |X_{s-}|^2)^p + \Lambda_{s-}^\beta \right] ds \\
&\leq \mathbb{E} \int_0^t C(s) \left[(1 + |X_{s \wedge \tau_M}|^2)^p + p \Lambda_{s \wedge \tau_M}^\beta \right] ds, \quad t \leq T.
\end{aligned}$$

By Gronwall's inequality,

$$\mathbb{E} \left[(1 + |X_{t \wedge \tau_M}|^2)^p + p \Lambda_{t \wedge \tau_M}^\beta \right] \leq \exp \left[\int_0^t C(s) ds \right] \left\{ (1 + |x|^2)^p + p i^\beta \right\}, \quad t \leq T.$$

Therefore, the solution to (1.3) is non-explosive on $[0, T]$.

The proof of the claim under the condition (2) is similar. Let $\lambda = \alpha c$. By Itô's formula, we have that

$$\begin{aligned}
&d \exp \left\{ e^{-\lambda t} (1 + |X_t|^2)^\alpha \right\} \\
&\leq \alpha e^{-\lambda t} (1 + |X_t|^2)^\alpha \exp \left\{ e^{-\lambda t} (1 + |X_t|^2)^\alpha \right\} \left\{ -\frac{\lambda}{\alpha} \right. \\
&\quad \left. + \frac{2\langle x, b(x, j, t) \rangle + (1 + 2\alpha(1 + |x|^2)^\alpha) |\sigma(x, j, t)|_{HS}^2}{1 + |x|^2} \right\} dt \\
&\quad + 2\alpha e^{-\lambda t} (1 + |X_t|^2)^{\alpha-1} \langle X_t, \sigma(X_t, \Lambda_t, t) dW_t \rangle, \quad t < \tau_M,
\end{aligned}$$

and

$$\begin{aligned}
e^{-\lambda t \wedge \tau_M} \Lambda_{t \wedge \tau_M}^\beta &= i^\beta + \int_0^{t \wedge \tau_M} e^{-\lambda s} \int_{\mathbb{R}^+} \left[(\Lambda_{s-} + h(X_{s-}, \Lambda_{s-}, z))^\beta - \Lambda_{s-}^\beta \right] \tilde{N}(dz, ds) \\
&\quad + \int_0^{t \wedge \tau_M} e^{-\lambda s} \sum_{k=1}^{\infty} \left[k^\beta - \Lambda_{s-}^\beta \right] q_{\Lambda_{s-k}}(X_{s-}) ds - \lambda \int_0^{t \wedge \tau_M} e^{-\lambda s} \Lambda_s^\beta ds.
\end{aligned}$$

Then, by (2.11) and $\lambda = \alpha c$, we obtain that

$$\begin{aligned}
&\mathbb{E} \left\{ \exp \left\{ e^{-\lambda t \wedge \tau_M} (1 + |X_{t \wedge \tau_M}|^2)^\alpha \right\} + \alpha e^{-\lambda t \wedge \tau_M} \Lambda_{t \wedge \tau_M}^\beta \right\} \\
&\leq \exp \{ (1 + |x|^2)^\alpha \} + \alpha i^\beta - \alpha \mathbb{E} \int_0^{t \wedge \tau_M} e^{-\lambda s} \Lambda_{s-}^\beta ds \\
&\quad + \mathbb{E} \left\{ \int_0^{t \wedge \tau_M} \alpha e^{-\lambda s} (1 + |X_s|^2)^\alpha \exp \left\{ e^{-\lambda s} (1 + |X_s|^2)^\alpha \right\} \left[-\frac{\lambda}{\alpha} \right. \right. \\
&\quad \left. \left. + \frac{2\langle X_s, b(X_s, \Lambda_{s-}, s) \rangle + (1 + 2\alpha(1 + |X_s|^2)^\alpha) |\sigma(X_s, \Lambda_{s-}, s)|_{HS}^2}{1 + |X_s|^2} \right. \right. \\
&\quad \left. \left. + \frac{\sum_{k=1}^{\infty} (k^\beta - \Lambda_{s-}^\beta) q_{\Lambda_{s-k}}(X_s)}{(1 + |X_s|^2)^\alpha \exp \{ e^{-\lambda s} (1 + |X_s|^2)^\alpha \}} \right] ds \right\} \\
&\leq \exp \{ (1 + |x|^2)^\alpha \} + \alpha i^\beta + (c-1) \alpha \mathbb{E} \int_0^{t \wedge \tau_M} e^{-\lambda s} \Lambda_{s-}^\beta ds.
\end{aligned}$$

By Gronwall's inequality, there is $C > 0$ which is independent of M such that

$$\mathbb{E} \left\{ \exp \left\{ e^{-\lambda t \wedge \tau_M} (1 + |X_{t \wedge \tau_M}|^2)^\alpha \right\} + \alpha e^{-\lambda t \wedge \tau_M} \Lambda_{t \wedge \tau_M}^\beta \right\} \leq C.$$

Therefore, the proof is completed. \square

Remark 2.1. *The conditions in (1) and (2) are an extension of that used in [17, Lemma 3.1], and they can be applied to the situation treated in [10, Theorem 2.3]. In fact, in that case, we can let $\beta = 1$, $p = 1$. Comparing with [10, Theorem 2.3], our conditions allow that $\{q_i(\cdot)\}_{i \in \mathbb{S}}$ are not necessary uniformly bounded on \mathbb{R}^d . However, it is a pity that $q_i(x)$ is at most linear growth for the variable i .*

We present the following example to illustrate our conditions on Q -matrix $Q(x)$.

Example 2.5. *Let $p \geq 1$, $\gamma > 2$, $q_{jk}(x) = \frac{j+|x|^p}{|k-j|^\gamma}$, $j \neq k$, $k, j \in \mathbb{S}$, and $C = \sum_{k=1}^\infty \frac{1}{k^\gamma}$. Then*

$$\begin{aligned} q_j(x) &= \left(\sum_{k \geq j+1}^\infty \frac{1}{|k-j|^\gamma} + \sum_{1 \leq k \leq j} \frac{1}{|k-j|^\gamma} \right) (j + |x|^p) \leq 2C(j + |x|^p), \\ q_j(x) &\geq (j + |x|^p) \sum_{k \geq j+1}^\infty \frac{1}{|k-j|^\gamma} \geq C(j + |x|^p), \quad j \in \mathbb{S}. \end{aligned}$$

For all $\beta \in [1, \gamma - 1)$, there exist positive constants C_β , $C_{\gamma, \beta}$ such that

$$\begin{aligned} \sum_{k=1}^\infty (k^\beta - j^\beta) q_{jk}(x) &\leq C_\beta \sum_{k \geq j+1} \frac{|k-j|^\beta + |k-j|^{j^\beta-1}}{|k-j|^\gamma} (j + |x|^p) \\ &\leq C_{\beta, \gamma} (j^\beta + |x|^p), \quad j \in \mathbb{S}. \end{aligned}$$

3 Strong Feller Property

In this section, we shall investigate the strong Feller property for (X_t, Λ_t) following the idea used in Theorem 2.1. Let f be a bounded measurable function on $\mathbb{R}^d \times \mathbb{S}$, $P_t f(x, i) = \mathbb{E} f(X_t^{x, i}, \Lambda_t^{x, i})$, and $P_t^K f(x, i) = \mathbb{E} f(X_t^K, \Lambda_t^K)$ with $(X_0^K, \Lambda_0^K) = (x, i)$. Denote the transition semigroup associated with (2.1) by $P_{t_0, t+t_0}^i$ (P_t^i if $t_0 = 0$), i.e.

$$P_{t_0, t+t_0}^i f(x) = \mathbb{E} f(x_t^{x, i, t_0}).$$

Theorem 3.1. *Assume that (H) holds, moreover for each $i \in \mathbb{S}$, q_i is locally bounded and the semigroup $\{P_t^i\}_{t \geq 0}$ generated by (2.1) ($t_0 = 0$) is strong Feller. Let $(x, i) \in \mathbb{R}^d \times \mathbb{S}$, $M \in \mathbb{S}$, $M > |x| + i$,*

$$\tau_M^{x, i} = \inf \left\{ t \geq 0 \mid |X_t^{x, i}| + \Lambda_t^{x, i} \geq M \right\}.$$

Assume that for each $x \in \mathbb{R}^d$, there exists $\delta > 0$ such that

$$\lim_{M \rightarrow \infty} \sup_{|y-x| \leq \delta} \mathbb{P} \left(\tau_M^{y, i} \leq t \right) = 0. \quad (3.1)$$

Then P_t is strong Feller for $t > 0$.

The discussion here begins with the solution to (2.3) (X_t^K, Λ_t^K) . But we will study (2.3) “path by path” of $\{N_t^K\}_{t \geq 0}$. Fix $T > 0$, $K > 0$. Let ω_2 be a path of the process N_t^K . Then ω_2 has finite jumps on $[0, T]$ at most. Denote that $\Delta\omega_2(t) = \omega_2(t) - \omega_2(t-)$. Let $(x_t^{(\omega_2)}, \lambda_t^{(\omega_2)})$ be the solution of

$$\begin{cases} dx_t^{(\omega_2)} = b(x_t^{(\omega_2)}, \lambda_t^{(\omega_2)}, t)dt + \sigma(x_t^{(\omega_2)}, \lambda_t^{(\omega_2)}, t)dW_t, & x_0^{(\omega_2)} = x, \\ \lambda_t^{(\omega_2)} = \sum_{s \leq t} h(x_s^{(\omega_2)}, \lambda_s^{(\omega_2)}, \Delta\omega_2(s)), & \lambda_0^{(\omega_2)} = i. \end{cases} \quad (3.2)$$

Let t_n be the n -th discontinuous point of ω_2 , and

$$m(\omega_2) = \sup_n \{t_n \leq T\}.$$

Lemma 3.2. *Assume that (H) holds. Let f be a bounded Borel measurable function on $\mathbb{R}^d \times \mathbb{S}$. For each ω_2 , there exists $g_m^{(\omega_2)} : \mathbb{R}^d \times \mathbb{S} \rightarrow \mathbb{R}$ which is bounded and independent of (x, i) such that*

$$\mathbb{E}f(x_T^{(\omega_2)}, \lambda_T^{(\omega_2)}) = P_{t_1 \wedge T}^i \left[g_m^{(\omega_2)}(\cdot, i) \right] (x).$$

Proof. We shall prove this lemma by induction. For $0 < T < t_1$, we have

$$\mathbb{E}f(x_T^{(\omega_2)}, \lambda_T^{(\omega_2)}) = \mathbb{E}f(x_T^{(\omega_2)}, i) = \mathbb{P}_T^i f(x).$$

If $T = t_1$, then

$$\mathbb{E}f(x_T^{(\omega_2)}, \lambda_T^{(\omega_2)}) = \mathbb{E}f(x_T^{(\omega_2)}, i + h(x_T^{(\omega_2)}, i, \Delta\omega_2(T))) = P_T^i \left[g_1^{(\omega_2)}(\cdot, i) \right] (x),$$

where $g^{(\omega_2)}(y, j) = f(y, j + h(y, j, \Delta\omega_2(t_1)))$, $(y, j) \in \mathbb{R}^d \times \mathbb{S}$. So, on $[0, t_1]$, there exists $g^{(\omega_2)} : \mathbb{R}^d \times \mathbb{S} \rightarrow \mathbb{R}$ which is bounded and independent of (x, i) such that

$$\mathbb{E}f(x_T^{(\omega_2)}, \lambda_T^{(\omega_2)}) = P_{t_1 \wedge T}^i \left[g^{(\omega_2)}(\cdot, i) \right] (x), \quad T \in (0, t_1].$$

Next, we assume that for $T \in [0, t_n]$, there is $g_n^{(\omega_2)} : \mathbb{R}^d \times \mathbb{S} \rightarrow \mathbb{R}$ which is independent of (x, i) such that

$$\mathbb{E}f(x_T^{(\omega_2)}, \lambda_T^{(\omega_2)}) = P_{t_1 \wedge T}^i \left[g_n^{(\omega_2)}(\cdot, i) \right] (x).$$

When $T \in (t_n, t_{n+1})$, due to the argument used in Lemma 2.3, we have

$$\begin{aligned} \mathbb{E}f(x_T^{(\omega_2)}, \lambda_T^{(\omega_2)}) &= \mathbb{E}f(x_{T-t_n+t_n}^{(\omega_2)}, \lambda_{t_n}^{(\omega_2)}) = \mathbb{E}f \left(x_{T-t_n}^{(x_{t_n}^{(\omega_2)}, \lambda_{t_n}^{(\omega_2)})}, \lambda_{t_n}^{(\omega_2)} \right) \\ &= \mathbb{E} \left\{ \mathbb{E} \left[f \left(x_{T-t_n}^{(x_{t_n}^{(\omega_2)}, \lambda_{t_n}^{(\omega_2)})}, \lambda_{t_n}^{(\omega_2)} \right) \mid (x_{t_n}^{(\omega_2)}, \lambda_{t_n}^{(\omega_2)}) \right] \right\} \\ &= \mathbb{E} \left\{ \left[\left(P_{t_n, T}^j f(\cdot, j) \right) (y) \right]_{(y, j) = (x_{t_n}^{(\omega_2)}, \lambda_{t_n}^{(\omega_2)})} \right\} \\ &\equiv \mathbb{E} \tilde{g}_n^{(\omega_2)}(x_{t_n}^{(\omega_2)}, \lambda_{t_n}^{(\omega_2)}). \end{aligned}$$

If $T = t_{n+1}$, then

$$\lambda_{t_{n+1}}^{(\omega_2)} = \lambda_{t_n}^{(\omega_2)} + h(x_{t_{n+1}}^{(\omega_2)}, \lambda_{t_n}^{(\omega_2)}, \Delta\omega_2(t_{n+1}))$$

$$= \lambda_{t_n}^{(\omega_2)} + h(x_{t_{n+1}-t_n}^{(\omega_2)}, \lambda_{t_n}^{(\omega_2)}, \Delta\omega_2(t_{n+1})).$$

Arguing as $T = t_1$, let

$$\tilde{g}_n^{(\omega_2)}(y, j) = P_{t_n, T}^j f(\cdot, j + h(\cdot, j, \Delta\omega_2(t_{n+1}))) (y).$$

Then

$$\mathbb{E}f(x_T^{(\omega_2)}, \lambda_T^{(\omega_2)}) = \mathbb{E}\tilde{g}_n^{(\omega_2)}(x_{t_n}^{(\omega_2)}, \lambda_{t_n}^{(\omega_2)}).$$

By induction, we prove the lemma. \square

Proof of Theorem 3.1

By (3.1), (X_t, Λ_t) is non-explosive. Let $K = \sup_{|x| \leq M} \sum_{k=1}^{M+1} q_k(x)$. According to (2.7), $(X_t, \Lambda_t) = (X_t^K, \Lambda_t^K)$, $t < \tau_M^{x, i}$. Let τ_n be the time of the n -th arrival of $N(K, \cdot)$, $m = \max\{n \mid \tau_n \leq t\}$. Then $\tau_n > 0$, \mathbb{P} -a.s. Let $\mathbb{E}_{\mathbb{P}_2}$ be the distribution of the process $\{N_t^K\}_{t \in [0, T]}$ on the path space. We have

$$\begin{aligned} \mathbb{E}f(X_t^K, \Lambda_t^K) &= \mathbb{E} \left\{ \mathbb{E} \left[f(X_t^K, \Lambda_t^K) \mid N_t^K \right] \right\} \\ &= \mathbb{E}_{\mathbb{P}_2} \left\{ \mathbb{E} \left[f(x_t^{(\omega_2)}, \lambda_t^{(\omega_2)}) \mid N_t^K = \omega_2 \right] \right\} \\ &= \mathbb{E}_{\mathbb{P}_2} P_{\tau_1 \wedge t}^i g_{m(\omega_2)}^{(\omega_2)}(x). \end{aligned}$$

Then by strong Feller property of P_t^i and the dominate convergence theorem, we obtain that

$$\begin{aligned} \lim_{y \rightarrow x} P_t^K f(y, i) &= \lim_{y \rightarrow x} \mathbb{E}f(X_t^K, \Lambda_t^K) = \lim_{y \rightarrow x} \mathbb{E}_{\mathbb{P}_2} P_{\tau_1 \wedge t}^i g_{m(\omega_2)}^{(\omega_2)}(x) \\ &= \mathbb{E}_{\mathbb{P}_2} \lim_{y \rightarrow x} P_{\tau_1 \wedge t}^i g_{m(\omega_2)}^{(\omega_2)}(y) = \mathbb{E}_{\mathbb{P}_2} P_{\tau_1(\omega_2) \wedge t}^i g_{m(\omega_2)}^{(\omega_2)}(x) \\ &= P_t^K f(x, i). \end{aligned} \quad (3.3)$$

On the other hand

$$\begin{aligned} |P_t f(x, i) - P_t^K f(x, i)| &\leq \mathbb{E} |f(X_t, \Lambda_t) - f(X_t^K, \Lambda_t^K)| \\ &\leq \mathbb{E} \left\{ |f(X_t, \Lambda_t) - f(X_t^K, \Lambda_t^K)| \mathbb{1}_{[\tau_M^{x, i} \leq t]} \right\} + \mathbb{E} \left\{ |f(X_t, \Lambda_t) - f(X_t^K, \Lambda_t^K)| \mathbb{1}_{[\tau_M^{x, i} > t]} \right\} \\ &\leq 2|f|_\infty \mathbb{P} \left(\tau_M^{x, i} \leq t \right). \end{aligned}$$

So

$$\begin{aligned} &|P_t f(x, i) - P_t f(y, i)| \\ &\leq |P_t f(x, i) - P_t^K f(x, i)| + |P_t f(x, i) - P_t^K f(x, i)| \\ &\quad + |P_t^K f(y, i) - P_t^K f(x, i)| \\ &\leq 2|f|_\infty \left[\mathbb{P} \left(\tau_M^{x, i} \leq t \right) + \mathbb{P} \left(\tau_M^{y, i} \leq t \right) \right] + |P_t^K f(y, i) - P_t^K f(x, i)|. \end{aligned} \quad (3.4)$$

Then it is stander to prove the strong Feller property of P_t via (3.4), (3.1), (3.3) and Lemma 3.2. \square

Remark 3.1. *This theorem partly generalizes [10, Theorem 3.2], and works well when the diffusion in each environment is degenerated.*

Combining the results in the previous section, it is easy to get that

Corollary 3.3. *Fix $T > 0$. Under the assumptions of Theorem 2.1 and Theorem 2.4, if for each $i \in \mathbb{S}$, the semigroup $\{P_t^i\}_{t \in (0, T]}$ generated by (2.1) ($t_0 = 0$) is strong Feller holds, then P_t has strong Feller property for $t > 0$.*

Proof. We only have to prove (3.1). Let $\tau_M^{x,i}$ as in Theorem 3.1. According to the proof of Theorem 2.4, there exists a locally bounded function $f : [0, T] \times \mathbb{R}^d \times \mathbb{S} \rightarrow [1, \infty)$ with

$$\lim_{M \rightarrow \infty} \left[\inf_{t \in [0, T], |y|+j \geq M} f(t, y, j) \right] = \infty$$

such that

$$\mathbb{E}f(t \wedge \tau_M, X_{t \wedge \tau_M}, \Lambda_{t \wedge \tau_M}) \leq Cf(0, x, i), \quad (x, i) \in \mathbb{R}^d \times \mathbb{S}$$

for some C independent of M, x, i . Then

$$\begin{aligned} \mathbb{P}(t \geq \tau_M^{x,i}) &\leq \mathbb{E} \frac{f(t \wedge \tau_M, X_{t \wedge \tau_M}, \Lambda_{t \wedge \tau_M})}{f(\tau_M, X_{\tau_M}, \Lambda_{\tau_M})} \mathbb{1}_{[t \geq \tau_M^{x,i}]} \\ &\leq \frac{\mathbb{E}f(t \wedge \tau_M, X_{t \wedge \tau_M}, \Lambda_{t \wedge \tau_M}) \mathbb{1}_{[t \geq \tau_M^{x,i}]}}{\inf_{t \in [0, T], |y|+j \geq M} f(t, y, j)} \\ &\leq \frac{Cf(0, x, i)}{\inf_{t \in [0, T], |y|+j \geq M} f(t, y, j)}, \quad (x, i) \in \mathbb{R}^d \times \mathbb{S}. \end{aligned}$$

Hence

$$\limsup_{M \rightarrow \infty} \sup_{|y-x| \leq 1} \mathbb{P}(t \geq \tau_M^{y,i}) \leq \lim_{M \rightarrow \infty} \sup_{|y-x| \leq 1} \frac{Cf(0, y, i)}{\inf_{t \in [0, T], |y|+j \geq M} f(t, y, j)} = 0.$$

□

Acknowledgements

The author would like to thank Professor Feng-Yu Wang and Jinghai Shao for their useful suggestions.

References

- [1] M.-F. Chen, From Markov chains to non-equilibrium particle systems, second edition, *Singapore: World Scientific*, (2004).
- [2] M. K. Ghosh, A. Arapostathis and S. I. Marcus, Optimal control of switching diffusions with application to flexible manufacturing systems, *SIAM J. Control Optim.*, **31** (1993), 1183–1204.
- [3] N. Ikeda and S. Watanabe, Stochastic differential equations and diffusion processes. Second edition. *North-Holland Publishing co. Amsterdam* (1989)
- [4] X. Mao and C. Yuan, Stochastic Differential Equations With Markovian Switching, *Imperial College Press, London*, (2006)

- [5] M. Pinsky and R. Pinsky, Transience/Recurrence and Central Limit Behavior for Diffusions in a Random Temporal Environment, *Ann. Probab.* **21** (1993), 433–452.
- [6] M. Pinsky and M. Scheutzow, Some remarks and examples concerning the transience and recurrence of random diffusions, *Ann. Inst. Henri. Poincaré*, **28** (1992), 519–536.
- [7] L. C. G. Rogers and D. Williams, Diffusions, Markov Processes, and Martingales: Volume 2, Itô Calculus. *Cambridge University Press; 2ed.* (2000)
- [8] J. Shao, Ergodicity of regime-switching diffusions in Wasserstein distances, *Stoch. Proc. Appl.* **125** (2015), 739–758. .
- [9] J. Shao, Criteria for transience and recurrence of regime-switching diffusion processes, *Electron. J. Probab.* **20** (2015), 1–15.
- [10] J. Shao, Strong solutions and strong Feller properties for regime-switching diffusion processes in an infinite state space. *SIAM J. Control Optim.* **53** (2015), 2462–2479.
- [11] J. Shao and F. B. Xi, Strong ergodicity of the regime-switching diffusion processes, *Stoch. Proc. Appl.* **123** (2013), 3903–3918.
- [12] J. Shao and F. B. Xi, Stability and recurrence of regime-switching diffusion processes, *SIAM J. Control Optim.* **52** (2014), 3496–3516.
- [13] A. V. Skorokhod, Asymptotic methods in the theory of stochastic differential equations, *American Mathematical Society, Providence, RI.* (1989)
- [14] F. B. Xi and G. Yin, Jump-diffusions with state-dependent switching: existence and uniqueness, Feller property, linearization, and uniform ergodicity, *Sci. China Math.* **54** (2011), 2651–2667.
- [15] F. B. Xi and G. Yin, The strong Feller property of switching jump-diffusion processes, *Statistics and probability letters*, **83** (2013), 761–767.
- [16] F. B. Xi and L. Q. Zhao, on the stability of diffusion processes with state-dependent switching, *Science in China Series A: Mathematics*, **49** (2006), 1258–1274.
- [17] L. Xie and X. Zhang, Sobolev differentiable flows of SDEs with local Sobolev and super-linear growth coefficients. arXiv:1407.5834
- [18] X. Zhang, Strong solutions of SDEs with singular drift and Sobolev diffusion coefficients, *Stoch. Proc. Appl.* **115** (2005), 1805–1818.
- [19] C. Zhu and G. Yin, On strong Feller, recurrence, and weak stabilization of regime-switching diffusions, *SIAM J. Control Optim.* **48** (2009), 2003–2031.
- [20] G. Yin and C. Zhu, Hybrid switching diffusions: properties and applications, Vol. 63, *Stochastic Modeling and Applied Probability, Springer, New York.* (2010)